Lectures on 'Bott + Tu' 080421

Ch I. DeRham Theory.

M^n smooth mfd

$\rightsquigarrow \Omega^k(M, \mathbb{R}) \simeq \Gamma(M, \Lambda^k T_M^*)$

e.g. $\omega = f(x)\, dx^1 \wedge dx^2$

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

$d\omega = \sum_{j \geq 3} \frac{\partial f}{\partial x^j} dx^j \wedge dx^1 \wedge dx^2$

$d^2 = 0 \qquad \left(\because \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \right)$

$\rightsquigarrow H^k(M, \mathbb{R}) = \left. \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)} \right|_{\Omega^k(M, \mathbb{R})}$

$\wedge : \Omega^k \times \Omega^\ell \longrightarrow \Omega^{k+\ell}$

$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega. \qquad (\because (fg)' = f'g + fg')$

decend $\rightsquigarrow (H^\bullet(M), \wedge)$

$(\Omega^\bullet(M), d, \wedge)$

diff. graded alg. (DGA).

Homological alg.

$(\mathcal{C}^\bullet, d)$ diff. cpx. $\Rightarrow H^\bullet(\mathcal{C}^\bullet) = \frac{\mathrm{Ker}\, d}{\mathrm{Im}\, d}$

$d^2 = 0$

$f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ chain map $\Rightarrow f^* : H^q(\mathcal{A}^\bullet) \to H^q(\mathcal{B}^\bullet)$

$f \circ d_A = d_B \circ f$

$0 \to \mathcal{A}^\bullet \xrightarrow{f} \mathcal{B}^\bullet \xrightarrow{g} \mathcal{C}^\bullet \to 0 \quad \Rightarrow$

short exact seq.

$$\cdots \to H^q(\mathcal{A}) \to H^q(\mathcal{B}) \to H^q(\mathcal{C}^\bullet) \xrightarrow{d^*} H^{q+1}(\mathcal{A}) \to \cdots$$

long exact sequence

$$[c] \in H^q(\mathcal{C}^\cdot) \Rightarrow c = g(b)$$
$$\Rightarrow g(db) = d(\underbrace{g\,b}_{c}) = 0$$
$$\Rightarrow db = f(a) \Rightarrow d^*[c] := [a]$$

$$\begin{array}{ccccc} \mathcal{A}^{q+1} & & \mathcal{B}^{q+1} & & \mathcal{C}^{q+1} \\ a & \longmapsto & db & \longmapsto & 0 \\ & & \uparrow & & \uparrow \\ & & b & \longmapsto & c \\ \mathcal{A}^q & & \mathcal{B}^q & & \mathcal{C}^q \end{array}$$

Category

Functor $\begin{pmatrix} F(1_A) = 1_{F(A)} \\ F(g\circ f) = F(g)\circ F(f) \end{pmatrix}$

$$((\text{mfd})) \xrightarrow[\text{contravar.}]{\Omega^\cdot(\cdot)} ((\text{diff. cpx})) \xrightarrow[\text{covar.}]{H^\cdot} ((\text{vector spaces}))$$

or $((\text{DGA}))$ or $((\text{graded alg.}))$

$\begin{cases} \text{objects} \\ \text{morphism:} \end{cases}$ smooth maps, chain maps, homomorphisms

Eg. $((\Omega^\cdot(M), d, \wedge))$

$((\Omega^\cdot_c(M), d, \wedge))$ diff form w/ cpt support.

Eg. $M = \mathbb{R}^1$

$$0 \to \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \longrightarrow 0$$
$$f(x) \longmapsto f'(x)\,dx$$

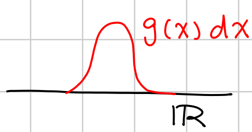
$H^0(\mathbb{R}) = \mathrm{Ker}(d|_{\Omega^0}) = \mathbb{R}$ const. fu.

$H^1(\mathbb{R}) = \dfrac{\Omega^1(\mathbb{R})}{\mathrm{Im}(d)} = 0 \quad \left(\because\ g(x)\,dx = d\left(\int_0^x g(u)\,du\right)\right)$

$$0 \longrightarrow \Omega^0_c(\mathbb{R}) \xrightarrow{d} \Omega^1_c(\mathbb{R}) \longrightarrow 0$$

$H^0_c(\mathbb{R}) = 0$

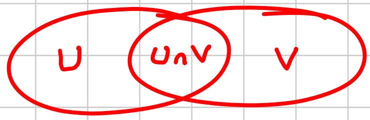
$H^1_c(\mathbb{R}) \xrightarrow[\cong]{\int} \mathbb{R}$



(Ex: injective).

Mayer-Vietoris Sequence

(paste together pieces)



$$M = U \cap V$$

mfd

$$M \longleftarrow U \amalg V \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U \cap V$$

$\downarrow$

(∵ ((DGA)) is linear)

DGA $\quad 0 \to \Omega^\bullet(M) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\partial_0^* - \partial_1^*} \Omega^\bullet(U \cap V) \to 0$

short exact seq. $\quad$ (∵ partition of 1)

$\downarrow$

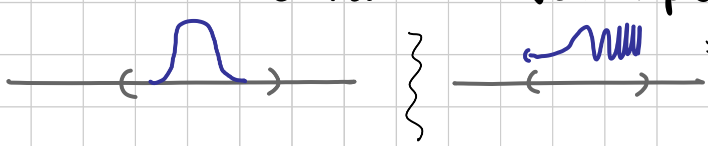
long exact seq.

$H^\bullet$

$$\begin{tikzcd} H^\bullet(M) \arrow[r] & H^\bullet(U) \oplus H^\bullet(V) \arrow[r] & H^\bullet(U \cap V) \arrow[ll, bend left, "\bullet + 1"] \end{tikzcd}$$

Eg. $\quad H^0(S^1) \simeq \mathbb{R}$
$\quad\quad H^1(S^1) \simeq \mathbb{R}$

For cpt support, $\Omega_c^\bullet$ $\begin{cases} \text{contravar. for proper maps} \\ \text{covariant for open inclusions} \end{cases}$



: Trouble for non-cpt supp. forms.

$$0 \leftarrow \Omega_c^\bullet(M) \xleftarrow[\text{sum}]{} \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) \xleftarrow[(-j_*,\, j_*)]{} \Omega_c^\bullet(U \cap V) \leftarrow 0$$

short exact seq.

$\Rightarrow$ long exact seq.

$$\begin{tikzcd} H_c^\bullet(M) \arrow[rr, bend right, "\bullet + 1"'] & H_c^\bullet(U) \oplus H_c^\bullet(V) \arrow[l] & H_c^\bullet(U \cap V) \arrow[l] \end{tikzcd}$$

Poincaré lemma (≡ Stokes' thm / integrat^{n} by part)

- $\omega = f(t)$ fu. on $\mathbb{R}$

$$f(t) - f(0) = \int_0^t \frac{df}{dt}\,dt = \int_0^t (df) = K\,d\,(\omega)$$

- $\omega = f(t)\,dt$ 1-form on $\mathbb{R}$

$$f(t)\,dt = d\left(\int_0^t f(u)\,du\right) = d\,K\,(\omega)$$

Define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \longrightarrow \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$

$$(\pi^*\phi)\cdot f(x,t) \longmapsto 0$$

$$(\pi^*\phi)\cdot f(x,t)\,dt \mapsto (\pi^*\phi)\cdot \int_0^t f\,dt$$

$$\begin{array}{c} \mathbb{R}^n \times \mathbb{R} \\ \pi \big\downarrow \big\uparrow s \ \text{0-section} \\ \mathbb{R}^n \end{array}$$

$1 - s^* \circ \pi^* = 0 \ : \Omega^\cdot(\mathbb{R}^n) \circlearrowleft \qquad (\because \pi \circ s = 1_{\mathbb{R}^q})$

$1 - \pi^* \circ s^* = \pm (dK \pm Kd) \ : \Omega^\cdot(\mathbb{R}^n \times \mathbb{R}) \circlearrowleft$

(K: homotopy operator)

$$\Rightarrow \quad H^q(\mathbb{R}^n \times \mathbb{R}) \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} H^q(\mathbb{R}^n) \qquad \cong$$

- $H^q(\mathbb{R}^n) = \mathbb{R}$ if $q=0$ and 0 otherwise.

- $H^q(M \times \mathbb{R}) \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} H^q(M) \qquad \cong$

- $f \sim g : M \to N \Rightarrow f^* = g^* : H^\cdot(N) \to H^\cdot(M)$
  (homotopic maps)

- $M \sim N \quad \Rightarrow \quad H^\cdot(M) \simeq H^\cdot(N)$
  homotopy equiv.

- $$H^q(S^n) = \begin{cases} \mathbb{R} & q = 0, n \\ 0 & \text{otherwise} \end{cases}$$

For cpt. support.

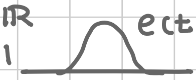
$$\begin{tikzcd} \Omega_c^\bullet(M\times\mathbb{R}) \arrow[d, shift right, "\pi_*"'] \\ \Omega_c^{\bullet-1}(M) \arrow[u, shift right, "e_* = \wedge e"'] \end{tikzcd}$$

integrat^n along fibers

(bump form)

$e(t):\mathbb{R}\to\mathbb{R}$

$\int \underbrace{e(t)\,dt}_{e} = 1$



$e(t)$

$$K:\Omega_c^q(\mathbb{R}^n\times\mathbb{R}) \longrightarrow \Omega_c^{q-1}(\mathbb{R}^n\times\mathbb{R})$$

$$\phi\cdot f(x,t) \longmapsto 0$$

$$\phi\cdot f(x,t)dt \mapsto \phi\cdot\left[\int_{-\infty}^t f - \left(\int_{-\infty}^t e\right)\cdot\int_{-\infty}^{\infty} f\right]$$

$$\pi_*\circ e_* - 1 = 0 : \Omega_c^\bullet(\mathbb{R}^n)\circlearrowleft$$

$$e_*\circ\pi_* - 1 = (-1)^q(dK-Kd): \Omega_c^\bullet(M\times\mathbb{R})\circlearrowleft$$

$$\Rightarrow \quad H_c^\bullet(M\times\mathbb{R}) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^{\bullet-1}(M)$$

- $H_c^q(\mathbb{R}^n) = \begin{cases}\mathbb{R} & q=n \\ 0 & \text{otherwise}\end{cases}$

- For vector bundle $\mathbb{R}^n \longrightarrow E \underset{s}{\overset{\pi}{\rightleftarrows}} M$

$$H^\bullet(E) \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} H^\bullet(M) \quad (\cong)$$

$$H_c^\bullet(E) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^{\bullet-n}(M) \quad (\cong)$$

Need E orientable VB ($\because$ integr. along fiber : π_*).

(Thom isom.) $\quad H_{cv}^\bullet(E) \xrightarrow[\cong]{\pi_*} H^{\bullet-n}(M)$

compact vertical co .

Recall $\qquad H^q(\mathbb{R}^n) \simeq H_c^{n-q}(\mathbb{R}^n)^*$

induces from $\qquad \Omega^q(\mathbb{R}^n) \times \Omega_c^{n-q}(\mathbb{R}^n) \xrightarrow{\int - \wedge -} \mathbb{R}$

Poincaré duality

$$H^q(M^n) \simeq H_c^{n-q}(M^n)^*$$

if M orientable (w/ finite good cover)

Reason:

(i) Integration $\quad \Omega^q(M^n) \times \Omega_c^{n-q}(M^n) \xrightarrow{\int - \wedge -} \mathbb{R}$

Or on any open $U \subset M$

$\overset{\text{Stokes}}{\Longrightarrow} \qquad H^q(U) \xrightarrow{\int} H_c^{n-q}(U)^*$

(ii) $\left.\begin{array}{l} M = U \cup V \\ \text{P.D. ✓ for } U, V, U\cap V \end{array}\right\} \Rightarrow$ P.D. ✓ for M

$$\begin{tikzcd}
\cdots \arrow[r] & (H^{q-1}(U)+H^{q-1}(V)) \arrow[r] \arrow[d, "\cong"'] & H^{q-1}(U\cap V) \arrow[r] \arrow[d, "\cong"'] & H^q(M) \arrow[r] \arrow[d] & (H^q(U)+H^q(V)) \arrow[r] \arrow[d, "\cong"'] & H^q(U\cap V) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & (H_c^{n-q+1}(U)^*+H_c^{n-q+1}(V)^*) \arrow[r] & H_c(U\cap V)^* \arrow[r] & H_c^{n-q}(M)^* \arrow[r] & (H_c^{n-q}(U)^*+H_c^{n-q}(V)^*) \arrow[r] & H_c^{n-q}(U\cap V)^* \arrow[r] & \cdots
\end{tikzcd}$$

5-lemma $\Rightarrow \quad H^q(M) \xrightarrow{\cong} H_c^{n-q}(M)^*$

(iii) M admits good cover $\{U_\alpha\}$

i.e. every $U_{\alpha_1\cdots\alpha_p} := U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ diffeo. to $\mathbb{R}^n$

(reason: via Riem. metric & small geod. (convex) balls)

(iv) MV for $(U_0 \cup \cdots \cup U_{p-1}) \cup U_p$. Induction $\Rightarrow$ ✓.

Remark: $\quad M$ mfd w/ finite good cover

$\Rightarrow \quad \dim H^q(M)$ & $\dim H_c^q(M) < \infty \quad$ (By similar arguments)

Poincaré dual $\quad S^k \hookrightarrow M^{n+k}$ closed submfd.

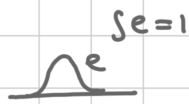
$\rightsquigarrow \quad P.D.[S] \in H^n(M)$

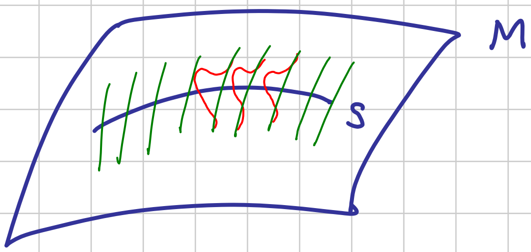
s.t.

$$\int_S \omega = \int_M \omega \wedge (P.D.[S]) \qquad \forall\, \omega \in H^k_c(M)$$

$\left(\text{reason: } \int_S (\ \ ) \in H^k_c(M)^* \simeq H^n(M)\right)$

e.g. $\quad \{0\} \subset \mathbb{R}^n \Rightarrow P.D.[0] = e(x)\, dx^1 \wedge \dots \wedge dx^n$





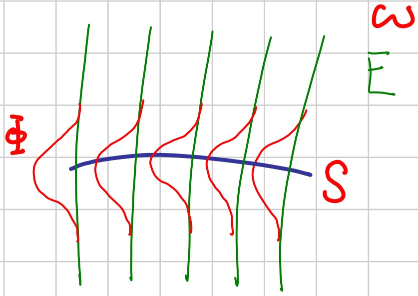
Support can be any open nbd of S (localization).

Recall Thom isom. $\quad \mathbb{R}^n \xrightarrow{\text{ori. bdl.}} E \xrightarrow{\pi} S$

$$H^\bullet(S) \xleftarrow[\cong]{\int_{E/M}} H^{\bullet+n}_{cv}(E)$$

$1 \longleftrightarrow \Phi \quad$ Thom class

$\omega \longleftrightarrow \pi^*\omega \wedge \Phi$

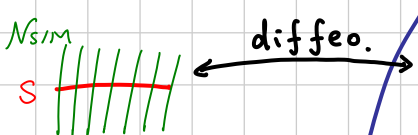


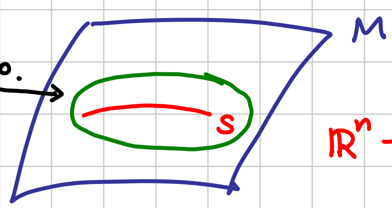
(reason: Partit$^{\underline{n}}$ of 1 & MV argument)

Back to $S^k \subset M^{n+k} \rightsquigarrow$ normal bdl./S

$$0 \to T_S \to T_M|_S \to N_{S/M} \to 0$$

tubular nbd thm:





Thom class $\longleftrightarrow$ Poincaré dual.

$\mathbb{R}^n \to N_{S/M} \to S \qquad PD[S] \in H^n(M)$

Künneth formula

$$\begin{array}{ccc} M\times F & \xrightarrow{p} & F \\ {\scriptstyle\pi}\downarrow & & \\ M & & \end{array} \quad \rightsquigarrow \quad H^\bullet(M)\otimes H^\bullet(F) \xrightarrow{\cong} H^\bullet(M\times F)$$

Pf: $M=\mathbb{R}^m$ ✓ (Poincaré lemma)

$M = U\cup V$ ✓ (MV / Partit^n of 1 / 5 lemma)

M has good cover ✓ (induction).

Similarly, for fiber bundle,

$$\begin{array}{ccc} F & \to & E \\ & & \downarrow \\ & & M \end{array} \qquad (\text{loc. } U\subset M \rightsquigarrow E|_U \simeq U\times F)$$

Leray-Hirsch If $\exists\, e_1,\dots,e_r\in H^\bullet(E)$, restr. $|_{\text{fiber}} \rightsquigarrow$ base for $H^\bullet(F)$

$$\Rightarrow H^\bullet(E)\simeq H^\bullet(M)\otimes\mathbb{R}\{e_1,\dots,e_r\}\simeq H^\bullet(M)\otimes H^\bullet(F)$$

(In general $\rightsquigarrow$ Leray spectral sequence.)

Ch 2 Čech-deRham Complex.

$M = U\cup V \qquad \mathcal{U}=\{U,V\}$ open cover

$$M \longleftarrow U\sqcup V \leftleftarrows U\cap V$$

$$\rightsquigarrow 0\to\Omega^\bullet(M)\longrightarrow \underbrace{\Omega^\bullet(U)+\Omega^\bullet(V)}_{C^0(\mathcal{U},\Omega^\bullet)} \xrightarrow{\delta} \underbrace{\Omega^\bullet(U\cap V)}_{C^1(\mathcal{U},\Omega^\bullet)}\to 0$$

short ex. seq. ($\rightsquigarrow$ long ex seq. on $H^\bullet$)

$$\begin{array}{c|ccc} q & \vdots & \vdots & \vdots \\ & \Omega^2(U)+\Omega^2(V) & \Omega^2(U\cap V) & 0 \\ d\uparrow\ \text{geometric} & \Omega^1(U)+\Omega^1(V) & \Omega^1(U\cap V) & 0 \\ & \Omega^0(U)+\Omega^0(V) & \Omega^0(U\cap V) & 0 \\ \hline & & \xrightarrow{\delta}\ \text{combinatoric} & p \end{array}$$

$K^{p,q}=C^p(\mathcal{U},\Omega^q)$ double cpx.

$$d^2=\delta^2=d\delta\pm\delta d=0$$

$$\Rightarrow D := d\pm\delta\ ,\quad D^2=0$$

δ : restriction

$D = d \pm \delta$

$$\begin{tikzcd}
 & q & & \\
\Omega^2(M) \arrow[u, "d"] \arrow[r, "r"] & \Omega^2(U)+\Omega^2(V) \arrow[u] \arrow[r, "\delta"] & \Omega^2(U\cap V) \arrow[u, "d"'] & 0 \\
\Omega^1(M) \arrow[u, "d"] \arrow[r] & \Omega^1(U)+\Omega^1(V) \arrow[u] \arrow[r] & \Omega^1(U\cap V) \arrow[u, "d"'] & 0 \\
\Omega^0(M) \arrow[u, "d"] \arrow[r] & \Omega^0(U)+\Omega^0(V) \arrow[u] \arrow[r] & \Omega^0(U\cap V) \arrow[u, "d"'] & 0 \\
 & & & p
\end{tikzcd}$$

$(\Omega^\bullet(M), d) \xrightarrow{r} (C^\bullet(\mathcal{U}, \Omega^\bullet), D)$ chain map

$\Rightarrow \quad H^\bullet(M) \xrightarrow{r^*} H_D($ —— " —— $)$

Claim: $\cong$

[δ has no coh. $\Rightarrow$ $H_d \xrightarrow{r} H_{D=d+\delta}$]

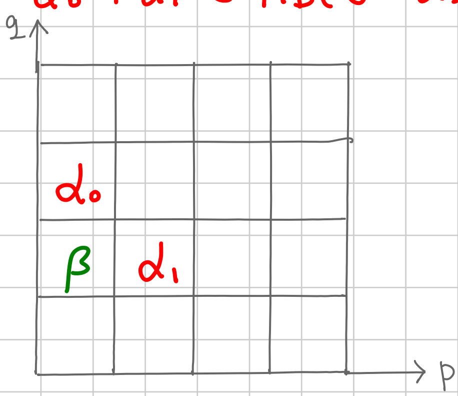
Reason

$$\begin{tikzcd}
 & q & & \\
\Omega^2(M) \arrow[u, "d"] \arrow[r, "r"] & \Omega^2(U)+\Omega^2(V) \arrow[u] \arrow[r, "\delta"] & \Omega^2(U\cap V) \arrow[u, "d"'] & 0 \\
\Omega^1(M) \arrow[u, "d"] \arrow[r] & \Omega^1(U)+\Omega^1(V) \arrow[u] \arrow[r] & \Omega^1(U\cap V) \arrow[u, "d"'] & 0 \\
\Omega^0(M) \arrow[u, "d"] \arrow[r] & \Omega^0(U)+\Omega^0(V) \arrow[u] \arrow[r] & \Omega^0(U\cap V) \arrow[u, "d"'] & 0 \\
 & & & p
\end{tikzcd}$$

$D = d \pm \delta$

$\xrightarrow{\delta}$

- $\alpha_0 + \alpha_1 \in H_D(C^\bullet(\mathcal{U}, \Omega^\bullet))$



$\delta\alpha_1 = 0$ (auto) $\Rightarrow$ $\alpha_1 = \delta\beta$ ($\because$ rows are exact). (MV)

$[\alpha_0 + \alpha_1] = [\underbrace{\alpha_0 + d\beta}_{\text{call } \phi}]$ ONLY has top comp.

$D\phi = 0 \iff \underbrace{\delta\phi = 0}_{\text{global form}}, \underbrace{d\phi = 0}_{\text{closed form}} \Rightarrow r^*$ surj.

Similar for r^* 1-1. ✓

Zig-Zag arguments.

General case M w/ countable open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$

$$M \leftarrow \coprod U_{\alpha_0} \leftleftarrows \coprod_{\alpha_0 < \alpha_1} U_{\alpha_0\alpha_1} \leftleftarrows \coprod_{\alpha_0<\alpha_1<\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \leftleftarrows \cdots\cdots$$

$$\Rightarrow \quad \Omega^\bullet(M) \xrightarrow{r} \prod \Omega^\bullet(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^\bullet(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \underbrace{\prod \Omega^\bullet(U_{\alpha_0\alpha_1\alpha_2})}_{K^{2,\bullet}} \xrightarrow{\delta} \cdots\cdots$$

$$\delta^2 = 0$$

- δ - exact (i.e. $H_\delta^\bullet = 0$)
- Čech-deRham cpx. $(C^\bullet(\mathcal{U}, \Omega^\bullet), d, \delta)$
- $r^*: H^\bullet_{DR}(M) \longrightarrow H_D(C^\bullet(\mathcal{U},\Omega^\bullet), \underbrace{d \pm \delta}_{D})$ $\cong$

reason:

$d \uparrow \Omega^q$ $\quad$ $d\uparrow$ $K^{pq} \xrightarrow{\delta}$

$\left.\begin{array}{l}\text{row seq.}\\ H_\delta^\bullet(K^{\bullet q}) = 0\end{array}\right\} \Rightarrow H^\bullet(\Omega^\bullet, d) \xrightarrow{\cong} H_D(K^\bullet)$

$d \uparrow$ $\Omega^q(M)$ $\quad$ $C^p(\mathcal{U}, \Omega^q)$

Čech cpx: $C^p(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta}$

$$0 \to \underbrace{\Omega^q(M)}_{\operatorname{Ker}\delta} \xrightarrow{r} \prod \Omega^q(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^q(U_{\alpha_0} \cap U_{\alpha_1})$$

$\prod \Omega^1(U_{\alpha_0\alpha_1\cdots})$

$d \uparrow$

$\prod \Omega^0(U_{\alpha_0\alpha_1\cdots})$

$\iota \uparrow$

$\prod \mathbb{R}$

const on each $U_{\alpha_0\alpha_1\cdots}$ (say conn.)

$(d=0)$

$\left.\begin{array}{l}\text{If } \mathcal{U} \text{ good cover}\\ \Rightarrow \text{Column exact}\\ \left(\begin{array}{l}\because U_{\alpha_0\cdots\alpha_p} \simeq \mathbb{R}^n\\ H^{>0}(\mathbb{R}^n) = 0\end{array}\right)\end{array}\right\} \overset{\text{same}}{\underset{\text{as before}}{\Longrightarrow}}$

$$\check{H}^\bullet(\mathcal{U}, \mathbb{R}) \triangleq H_\delta(C^\bullet(\mathcal{U}, \mathbb{R}))$$
$$\simeq H_D(C^\bullet(\mathcal{U}, \Omega^\bullet))$$
$$\simeq H^\bullet_{DR}(M) \qquad ☆$$

Presheaf

eg. $U \subset M \rightsquigarrow \Omega^q(U)$

eg. $U \subset M \rightsquigarrow \mathbb{Z}(U)$ or $\mathbb{R}(U)$ loc. const. fu. on U.

$\mathcal{F}: \quad U \overset{\text{open}}{\subset} M \rightsquigarrow \mathcal{F}(U)$ Abelian group

$i_U^V : V \hookrightarrow U \rightsquigarrow \rho_V^U := \mathcal{F}(i_U^V) : \mathcal{F}(U) \to \mathcal{F}(V)$ 'restrict^n'

s.t. $\mathcal{F}(i_V^V) = id.$ & $\mathcal{F}(i_V^W)\mathcal{F}(i_U^V) = \mathcal{F}(i_U^W)$

i.e. Functor $\mathcal{F}: ((\text{Top on } M)) \longrightarrow ((\text{Abelian gp.}))$

Take open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of M.

$\rightsquigarrow \quad C^p(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0 \cdots \alpha_p})$

δ (altern. restr.) $\qquad \delta^2 = 0$

$\rightsquigarrow \quad H^\bullet(\mathcal{U}, \mathcal{F}) := H^\bullet(C^\bullet(\mathcal{U}, \mathcal{F}), \delta)$

Čech cohomology

$$\check{H}^\bullet(X, \mathcal{F}) := \underset{\substack{\longrightarrow \\ \mathcal{U}}}{\lim} H^\bullet(\mathcal{U}, \mathcal{F})$$

wrt refinement

Eg. $\quad \check{H}^\bullet(M, \mathbb{R}) \simeq H^\bullet_{DR}(M)$

Euler class of oriented sphere bundle

$S^n \to E \xrightarrow{\pi} M$ Sphere bdl, i.e. str. gp. $\mathrm{Diff}(S^n)$

e.g. VB $\mathbb{R}^{n+1} \longrightarrow V \longrightarrow M$ (str. gp. $GL(n+1,\mathbb{R})$, or $O(n+1)$)

fiberwise unit sphere $\rightsquigarrow$ sphere bdl.

(i.e. $O(n+1) \subseteq \mathrm{Diff}(S^n)$)

Does sphere bdl always come from VB ?

Need $O(n+1) \hookrightarrow \mathrm{Diff}(S^n)$ homotopy equiv.

Yes, if $n \leqslant 3$.
No for higher n.

Recall Leray-Hirsch $F \longrightarrow E \longrightarrow M$

IF $\exists\, e_1, \dots, e_r \in H^\bullet(E)$, restr. $|_{\text{fiber}} \rightsquigarrow$ basis for $H^\bullet(F)$

$$\Rightarrow H^\bullet(E) = H^\bullet(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \simeq H^\bullet(M) \otimes H^\bullet(F)$$

- $e_1 = 1 \in H^0(E)$ always there.
- Simplest case : $F = S^n$, $H^{\neq 0,n}(S^n) = 0$.
  i.e. sphere bundle.

Want global angular form $\psi \in \Omega^n(E)$ s.t.

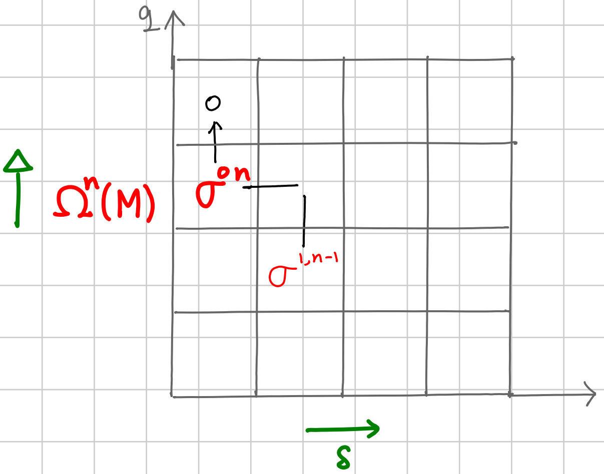
- $\psi|_{\text{fiber}}$ vol. form on fiber S^n
- $d\psi = -\pi^* e$

If $d\psi = 0$ then $H^\bullet(E) \cong H^\bullet(M) \otimes H^\bullet(S^n)$.

Construction: Choose good cover $\mathcal{U} = \{U_\alpha\}$ of M

$E|_{U_\alpha} \simeq U_\alpha \times S^n \rightsquigarrow [\sigma_\alpha] \in H^n(E_{U_\alpha})$ ✓

$\rightsquigarrow \sigma^{0,n} = \{\sigma_\alpha\} \in C^0(\mathcal{U}, \Omega^n)$



$(\delta \sigma^{0,n})_{\alpha\beta} = \sigma_\beta - \sigma_\alpha \in \Omega^n(E_{U_{\alpha\beta}})$

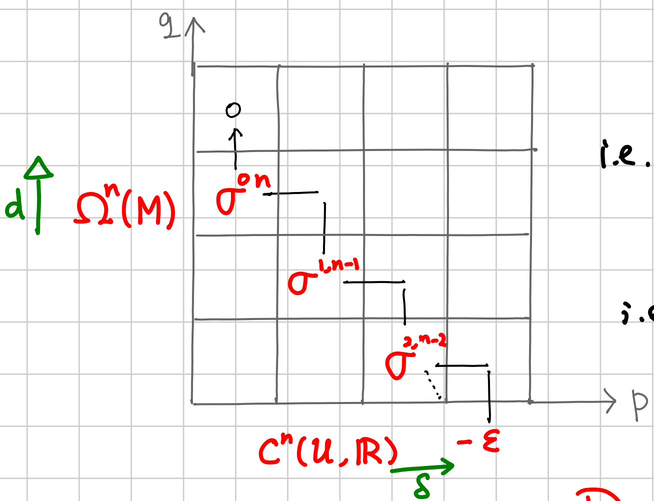
$= d(\sigma_{\alpha\beta})$

$\Updownarrow$

iff $[\sigma_\beta] = [\sigma_\alpha] \in H^n(E_{U_{\alpha\beta}})$

i.e. ori. sphere bdl.

Write $\sigma^{1,n-1} = \{\sigma_{\alpha\beta}\}$



$H^{<n}(S^n) = 0$ (except H^0)

i.e. Every closed k $(<n)$ form on $E_{U_{\alpha_0\alpha_1\cdots}}$ is exact.

i.e. continue $\rightsquigarrow$

$\sigma^{0,n} + \sigma^{1,n-1} + \dots + \sigma^{n,0} =: \sigma$

$D\sigma = \delta \sigma^{n,0} = i(-\varepsilon)$

$\exists\, e(E) \in \underbrace{H^{n+1}(\mathcal{U}, \mathbb{R})}_{\check{H}^{n+1}(M)}$ ($\because$ Good Cover)

$\rightsquigarrow$ Euler class for oriented sphere bundles

- $\exists$ global angular form on $S^n \to E \to M$ $\Leftrightarrow$ $\begin{cases} E\text{: orientable } + \\ e(E) = 0 \end{cases}$
- $\exists$ section $\Rightarrow$ $e(E) = 0$

$$S^n \longrightarrow E \longrightarrow M$$

$\exists$ explicit way to write down (i) global angular form ψ on E & (ii) Euler form e on M

(Modern way: Mathai-Quillen formula)

Eg S^1-bundle (section 6) For simplicity, assume

E = unit sphere of $(\mathbb{R}^2 \longrightarrow (V, \langle\ \rangle) \longrightarrow M)$.

- Choose $\{U_\alpha\}$ good cover s.t. $E_{U_\alpha} \simeq U_\alpha \times \underset{\theta_\alpha}{S^1}$
- On $\pi^{-1}(U_{\alpha\beta})$, $\theta_\beta - \theta_\alpha = \pi^* \varphi_{\alpha\beta}$, $\varphi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{R}$ unique up to $2\pi\,\mathbb{Z}$
- On $U_{\alpha\beta\gamma}$, $\underbrace{\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma}}_{\varepsilon_{\alpha\beta\gamma} \cdot 2\pi} \equiv 0 \pmod{2\pi}$

$$e(E) = [\varepsilon_{\alpha\beta\gamma}] \in \check{H}^2(M, \mathbb{Z})$$

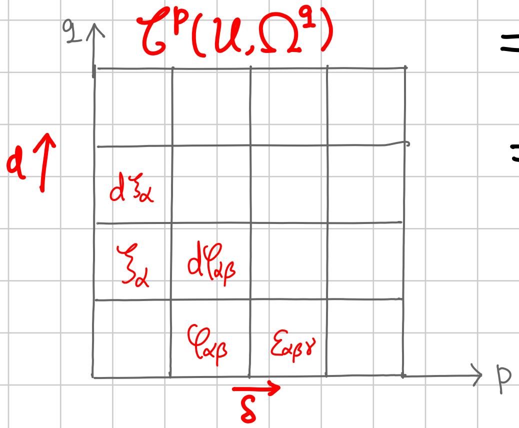
How about diff. form rep. $e(E)$?

$\varphi_{\alpha\beta} \in \Omega^0(U_{\alpha\beta})$, $\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} \equiv 0 \pmod{\overset{2\pi \equiv 1}{2\pi}}$

$\Rightarrow \underbrace{d\varphi_{\alpha\beta}}_{= \xi_\beta - \xi_\alpha} \in \Omega^1(U_{\alpha\beta})$, $d\varphi_{\alpha\beta} + d\varphi_{\beta\gamma} - d\varphi_{\alpha\gamma} = 0$

$= \xi_\beta - \xi_\alpha$ $\exists\ \xi_\alpha \in \Omega^1(U_\alpha)$ ($\because$ good cover)

(explicitly, if $\{\rho_\alpha : U_\alpha \to \mathbb{R}\}$ partition of 1, then $\xi_\alpha = \sum_\gamma \rho_\gamma\, d\varphi_{\alpha\gamma}$)



$\Rightarrow d\xi_\beta - d\xi_\alpha = d(d\varphi_{\alpha\beta}) = 0$

$\Rightarrow e \in \Omega^2(M)$ w/ $e|_{U_\alpha} = d\xi_\alpha$

- $$\begin{cases} \theta_\beta - \theta_\alpha = \pi^* \varphi_{\alpha\beta} & \text{on} \quad \pi^{-1}(U_{\alpha\beta}) = E_{U_{\alpha\beta}} \\ \xi_\beta - \xi_\alpha = d\varphi_{\alpha\beta} & \text{on} \quad U_{\alpha\beta} \end{cases}$$

$$\Rightarrow \quad d\theta_\beta - \pi^*\xi_\beta = d\theta_\alpha - \pi^*\xi_\alpha \quad \text{on} \quad \pi^{-1}U_{\alpha\beta}$$

patch $\rightsquigarrow$ $\psi \in \Omega^1(E)$ $\quad \begin{cases} \psi|_{S^1_{\text{fiber}}} = d\theta \\ d\psi = -\pi^* e \end{cases}$

global angular form.

- If describe $E = \bigcup_\alpha U_\alpha \times S^1$ w/ gluing

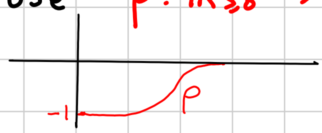
$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \begin{array}{c} SO(2) \\ \wr\rvert \\ U(1) \end{array} \quad \begin{array}{c} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ \updownarrow \\ e^{i\theta} \end{array} \quad (\leq \mathrm{Diff}^+(S^1))$$

$$\Rightarrow \quad \theta_\alpha - \theta_\beta = \pi^* \underbrace{\log g_{\alpha\beta}}_{-\varphi_{\alpha\beta}}$$

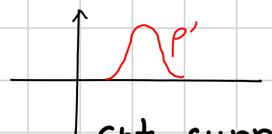
$$\Rightarrow \quad e(E) = \frac{i}{2\pi} \sum_\gamma d(\rho_\gamma \, d\log g_{\gamma\alpha}) \quad \text{on} \quad U_\alpha$$

- If
$$\begin{array}{ccccccc} \theta & \in & S^1 & \longrightarrow & E & \longrightarrow & M \\ & & \cap\rvert & & \cap\rvert & & \| \\ (r,\theta) & \in & \mathbb{R}^2 & \longrightarrow & V & \longrightarrow & M \end{array}$$

Choose $\rho : \mathbb{R}_{\geq 0} \longrightarrow [-1, 1]$



$\rho'(r)$:



$\int \rho' = 1$

cpt supp away from 0

$$\Phi := d(\rho \cdot \psi) \in \Omega^2_{cv}(V)$$

$$d\Phi = 0 \,, \quad \int_{V_{\text{fiber}}} \Phi = \int d\rho \wedge d\theta = 1$$

Thom class

- $$\Phi|_{\substack{\text{zero secton} \\ M}} = e \in H^2(M) \qquad (\because \rho'(0) = 0, \ \rho(0) = -1)$$

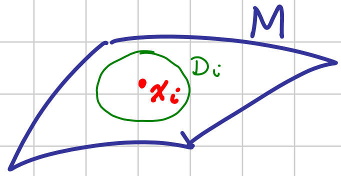
Similar in higher dim.

• $S^{k-1} \to E \xrightarrow{\pi} M^k$; $e(E) \in H^k(M) \simeq \mathbb{Z}$ Euler number

$\underset{(\text{cpt oriented})}{}$

$\forall$ section s over $\underbrace{M \setminus \{x_1, \dots, x_q\}}$

(i.e. $\pi \circ s = 1_M$)



$$\mathbb{Z} \ni \int_M e = \int_{M \setminus \cup D_i} \underbrace{e}_{s^* \underbrace{\pi^* e}_{-d\psi}}$$

$$\overset{\text{Stokes}}{=} \sum_i \int_{\partial D_i} s^* \psi$$

small $D_i \Rightarrow E_{D_i} \simeq D_i^k \times S^{k-1}$

(choose e supp away from x_i's)

$\because$ $\text{supp}(e) \subseteq U_{\alpha_0 \alpha_1 \cdots \alpha_{k-1}}$

$d\psi|_{E_{D_i}} = 0$, $\psi|_{\text{fiber}}$: generator of $\underbrace{H^{k-1}(S^{k-1})}_{\sigma}$

$\Rightarrow \psi - \sigma = d\tau$ (via product $E = D \times S$)

$$= \sum_i \underbrace{\int_{\partial D_i} s^* \sigma}_{\deg(s : \underbrace{\partial D_i}_{S^{k-1}} \to S^{k-1})} \Leftarrow$$

$\Rightarrow$ Thm $\int_M e = \sum_i (\text{loc. deg. of } s \text{ at } x_i)$.

Thm $\mathbb{R}^k \longrightarrow V \overset{s}{\rightleftarrows} M$ (ori., cpt.)

$\Rightarrow e(V) = \text{P.D.}\{s = 0\} \in H^k(M)$

• Lemma : $M \overset{\text{diag}}{\hookrightarrow} \Delta \subset M \times M$ $\Rightarrow$ $N_{\Delta/M\times M} \simeq T_M$

$x \longmapsto (x, x)$

Lemma $\underbrace{\text{P.D.}[\Delta]}_{\sum_i (-1)^{|\omega_i|} \omega_i(x) \wedge \omega^i(y)} \in H^n(M^n \times M^n)$

ω_i 's any base of $H^\bullet(M)$

ω^i 's dual base wrt $\int (\ ) \wedge (\ )$

i.e. $\int_{x \in M} \omega_i(x) \wedge \omega^j(x) = \delta_i^j$

Compute $\int_M e(T_M) = \int_\Delta e(N_{\Delta/M\times M}) = \int_\Delta (\text{P.D.}[\Delta])|_\Delta$

$$= \int_{M \simeq \Delta} \sum_i (-1)^{|\omega_i|} \omega_i(x) \wedge \omega^i(x) = \sum_i (-1)^{|\omega_i|}$$

$$= \sum (-1)^q \dim H^q(M) = \chi(M)$$

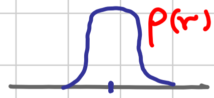
Hopf index thm $\chi(M) =$ Sum of index of any vector field.

Recap of chapter 2.

1) $\mathbb{R}^n$ vol. form / orientation

$\rho = f(x)\, dx^1 \wedge \cdots \wedge dx^n$ nonvanishing at every point x

Cohomological, only need $\int_{\mathbb{R}^n} \rho \neq 0 \rightsquigarrow \rho \in \Omega^n_c(\mathbb{R}^n)$

$f(x) = \rho(|x|)$ 

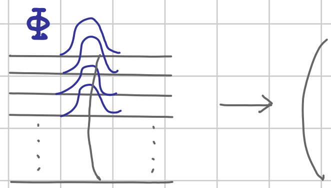
$[\rho] \in H^n_c(\mathbb{R}^n) \simeq H^n(S^n) \simeq \mathbb{R}$

- $\mathbb{R}^n \underset{\text{open}}{\subseteq} M^n \rightsquigarrow \quad H^n(M) \simeq \mathbb{R}, \quad \forall M$ cpt, oriented.
- Poincaré duality $\quad H^k \otimes H^{n-k} \to H^n \simeq \mathbb{R}$ perfect pairing

---

2) Vector bundle $\quad \mathbb{R}^n \longrightarrow E \xrightarrow{\pi} X$

Thom isom. $\quad H^{\bullet+n}_{cv}(E) \underset{\cong}{\xrightarrow{\int}} H^\bullet(M)$
(need orientability) $\quad \Phi \longleftrightarrow 1$



The way ρ is put in E to form $\Phi \rightsquigarrow$

Euler class $\quad e(E) = \Phi\big|_{\text{zero section}} \in H^n(M)$

(E, $\mathbb{R}^n$-bdl $\rightsquigarrow S^{n-1}$-bdl. e for any sphere bundle).

(E, $\mathbb{C}^n$-bdl $\rightsquigarrow$ Chern classes $\in H^{ev}(M)$.)

(Explicit $e(E)$ as Čech cocycle and as deRham form)

$$\mathbb{R}^n \to E \underset{s}{\rightleftarrows} M^n \Rightarrow \quad e(E) \in H^n(M^n) \overset{\int}{\simeq} \mathbb{R}$$

$$\int_M e(E) = \#\{s=0\} \qquad (\because \text{Stokes/Residue})$$

Recall $\mathbb{R}^n$, $\rho = \rho(r) r^{n-1} dr\, d\theta$ $\qquad \psi = d\theta \in \Omega^{n-1}(S^{n-1})$ angular form (NOT exact)

$= d(\tilde{\rho}(r)\, \psi)$

On
$$\begin{array}{ccccc} \mathbb{R}^n & \longrightarrow & E & \xrightarrow{\pi_E} & X \\ \cup\!\!| & & \cup\!\!| & & \| \\ S^{n-1} & \longrightarrow & S(E) & \xrightarrow{\pi_S} & X \end{array}$$

$\Phi = d(\tilde{\rho}(r)\, \psi)$ $\qquad \tilde{\rho}'(0) = 0$, $\tilde{\rho}(0) = -1$

$\pi_S^* e = -d\psi \quad \exists\, \psi \in \Omega^{n-1}(S(E))$

3) $M^m \subset X^{m+n}$. Consider $\int_M : H^m(X) \longrightarrow \mathbb{R}$

i.e. $\int_M \in H^m(X)^* \overset{P.D.}{\simeq} H^n(X)$ rep. by some diff form, called Poincaré dual, i.e. $P.D.[M] \in \Omega^n(X)$.

$$\int_M \varphi \simeq \int_X \varphi \wedge P.D.[M] \qquad \forall\, \varphi \in \Omega^m(X)$$

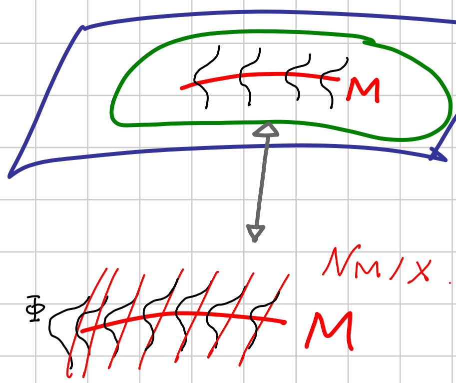
Which form? (like Thom isomorphism)

$$\mathrm{nbd}(M \subset X) \simeq N_{M/X}$$

Poincaré dual. $\quad P.D.[M] \in H^n(X)$

$\uparrow \qquad \uparrow$

Thom class for normal bdl. $\quad \Phi(N_{M/X}) \in H^n_c(N_{M/X})$



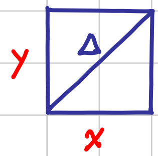
4) $M \xrightarrow[\text{diagonal}]{\Delta} M \times M =: X$

$$\mathrm{nbd}(M \subset X) \cong N_{\Delta/M\times M} \cong T_M$$

$$PD[M]\big|_\Delta \simeq \Phi(N)\big|_\Delta \simeq e(N) = e(T_M)$$

$$\underbrace{\int_\Delta PD[M]\big|_\Delta}_{\Sigma(-1)^i b_i(M)} = \int_M e(T_M)$$

$\frac{M}{X}$



Diag: $x+y$

Off-diag. $\delta x - \delta y = 2\delta x - \underbrace{\delta(x+y)}_{=0\ \in T_X/T_M}$

normal $\quad N_{M/X} \simeq T_M$

$N_{M/X} = T_X/T_M$

$$PD[\Delta] = \Phi(N_{\Delta/M\times M}) \sim \prod_{i=1}^{n} d(x^i - y^i) = \sum (-1)^{|I|} \underbrace{dx^I}_{\omega_I(x)} \wedge \underbrace{(\pm\, dy^{n-I})}_{\omega^I(y)}$$

$\omega_I(x) \wedge \omega^I(x) = dx^1 \wedge \cdots \wedge dx^n$

In general,

$$P.D.[\Delta] = \sum_i (-1)^{|\omega_i|} \omega_i(x) \wedge \omega^i(y) \in H^n(M_x \times M_y)$$

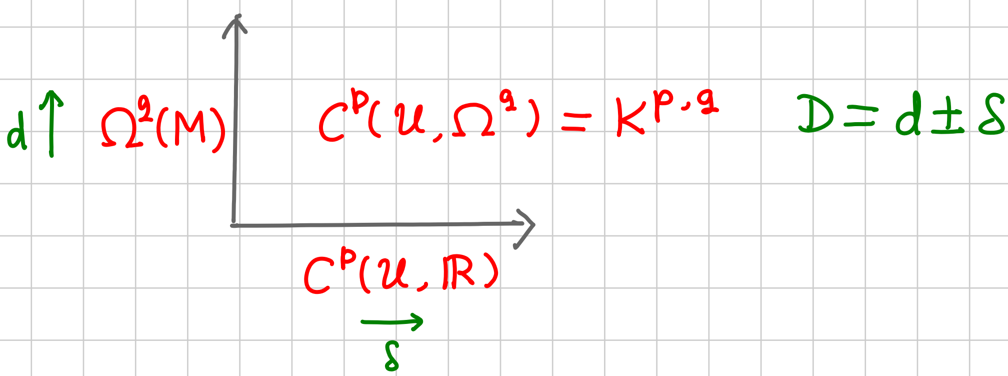
w/ $\{\omega_i\}$, $\{\omega^i\}$ dual bases for $H^\bullet(M)$.

$$\Rightarrow \int_\Delta P.D.[\Delta]|_\Delta = \sum (-1)^i b_i(M) =: \chi(M)$$

Cor. $\chi(M) = \#$ zero of any vector field.

§ Ch3. Spectral sequence

Recall



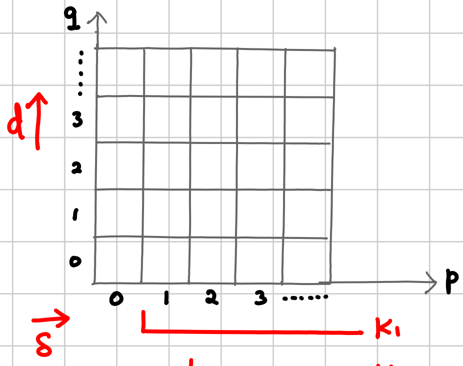
Double cpx.

- If rows are exact, then $H_d^\bullet(\Omega^\bullet(M)) = H_D^\bullet(K^{\bullet\bullet})$
- More generally, if $H_\delta H_d(K)$ has entries only in one row, then

$$H_\delta H_d(K) \simeq H_D$$

- Most general $\rightsquigarrow$ spectral sequence (powerful tools)

Double cpx $K = \bigoplus K^{p,q}$, d, δ



$\rightsquigarrow$ filtered cpx. $K_p = \bigoplus_{\substack{i \geq p \\ q \geq 0}} K^{i,q} = K^{\geq p, *}$

$K = K_0 \supset K_1 \supset K_2 \supset \cdots\cdots$ w/ $D = d \pm \delta$

$\rightsquigarrow$ exact couple $A = \bigoplus K_p$ & $B = \bigoplus \frac{K_p}{K_{p+1}}$ w/ D

$$0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0$$
$$(K_{p+1} \hookrightarrow K_p)$$

w/ long exact seq.

$$\begin{tikzcd} H(A) \arrow[rr, "i"] & & H(A) \arrow[dl, "j"] \\ & H(B) \arrow[ul, "k"] & \end{tikzcd}$$

$d = j \circ k$

$d^2 = 0$

Will get
$$E_1 = H_d(K)$$
$$E_2 = H_\delta H_d(K)$$
$$\Downarrow \qquad \vdots$$
$$E_\infty = H_D(K)$$

Exact couples

$$\begin{tikzcd} A \arrow[rr, "i"] & \text{exact} & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd} \Rightarrow \begin{array}{l} d = j \circ k : B \longrightarrow B \\ d^2 = j\,(\cancel{k\,j})\,k = 0 \end{array}$$

$\Rightarrow$

$$\begin{tikzcd} A' \arrow[rr, "i'"] & \text{exact} & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd} = \begin{tikzcd} i(A) \arrow[rr] & & i(A) \arrow[dl] \\ & H(B) \arrow[ul] & \end{tikzcd}$$

w/ $i' = i$ $\qquad k' = k$

$j'(ia) = [ja]$

$\Longrightarrow$ another exact couple

Spectral seq. for filtered complex.

(K, D) diff cpx. (i.e. $D^2 = 0$)

$K = K_0 \supset K_1 \supset K_2 \supset \cdots\cdots$ filtered cpx. (i.e. $DK_i \subseteq K_i$)

(extend: $K_{-\ell} = K_0 \ \forall -\ell < 0$)

$\rightsquigarrow$ graded cpx. $Gr.K = \bigoplus_{p=0}^{\infty} \frac{K_p}{K_{p+1}}$ w/ D

$\rightsquigarrow$ short exact sequence

$$0 \longrightarrow \underbrace{A}_{\bigoplus_p K_p} \xrightarrow{\ i\ } \underbrace{A}_{\bigoplus_p K_p} \xrightarrow{\ j\ } \underbrace{B}_{Gr.K} \longrightarrow 0$$

i.e. $i : K_{p+1} \hookrightarrow K_p$

$\Longrightarrow$ long exact seq.

$$\cdots\cdots \longrightarrow H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \longrightarrow \cdots\cdots$$

i.e. exact couple

$$\begin{tikzcd} H(A) \arrow[rr, "i_1"] & & H(A) \arrow[dl, "j_1"] \\ & H(B) \arrow[ul, "k_1"] & \end{tikzcd} =: \begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

$\Longrightarrow \forall\ r = 1, 2, 3, \cdots\cdots$

$$\begin{tikzcd} A_r \arrow[rr, "i_r"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd}$$

ex. couple

Usually write $E_r = B_r$,

$E_1 = H(B)$ $\quad d_1 = j_1 \circ k_1$

$E_2 = H(E_1)$ $\quad d_2 = j_2 \circ k_2$ $\quad , \cdots\cdots$

(E_r, d_r) : spectral seq.

If filtration on K has finite length ℓ

$\Longrightarrow$ i_ℓ inclusion ($\Rightarrow$ $k_\ell = 0$)

$\Longrightarrow$ $A_{\ell+1} = A_{\ell+2} = \cdots =: A_\infty$

$B_{\ell+1} = B_{\ell+2} = \cdots =: B_\infty$

$$\begin{tikzcd} A_{\ell+} \arrow[r, hook] & A_{\ell+} \arrow[dl] \\ B_{\ell+} \arrow[u, dashed, "0"] & \end{tikzcd}$$

i.e. (E_r, d_r) cgt at $E_{\ell+1}$ term.

Thm. $K = \bigoplus_{n\in\mathbb{Z}} K^n$ graded filt. cpx. w/ filtratⁿ. K_p (finite length $\forall n$)

$$0 \to \oplus K_{p+1} \to \oplus K_p \to \oplus \frac{K_p}{K_{p+1}} \to 0$$

s.s. $(E_r, d_r) \Rightarrow H_D^\bullet(K)$

(i.e. E_∞ & $H_D^\bullet(K)$ equal as 'graded' groups)

Given $K = \bigoplus_{p,q\geq 0} K^{p,q}$ double cpx

e.g. $F \longrightarrow E \xrightarrow{\pi} M$ fiber bdl.

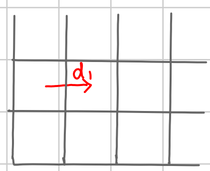
$\mathcal{U} = \{U_\alpha\}$ good cover

$$K^{p,q} = C^p(\pi^{-1}\mathcal{U}, \Omega^q) = \prod_{\alpha_0<\cdots<\alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0\cdots\alpha_p}), \quad d, \delta$$

$\rightsquigarrow$ $A = \oplus K_p$ $\quad K_p = K^{\geq p, *}$

$B = \oplus K_p/K_{p+1}$ $\quad$ (D on B $= (-1)^p d$) $\quad d_0 = \pm d$

$\Rightarrow$ $E_1 = H_d(K)$, $\quad d_1 = j_1 k_1 = \delta$

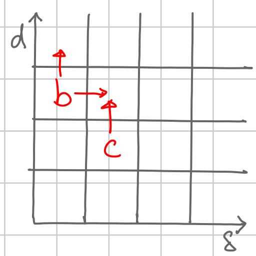


$\Rightarrow$ $E_2 = H_\delta H_d(K)$, $d_2 = ?$

$b \in E_2 \Rightarrow d_1 b = 0$

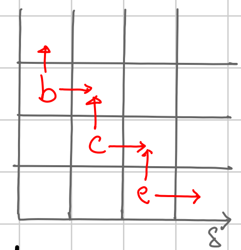
i.e. $db = 0$ & $\delta b = \pm dc$

$d_2 b = \delta c$



$\Rightarrow$ E_3 w/ $d_3 b = \delta c$

etc.



Thm. $Gr.\, H^n_D(K) = \bigoplus_{p+q=n} E_\infty^{p,q}$ (lose torsion inform).

For fiber bdl. eg. $E_1^{p,q} = C^p(\mathcal{U}, \mathcal{H}^q)$

$E_2^{p,q} = H^p_\delta(\mathcal{U}, \mathcal{H}^q)$

$\simeq H^p(M) \otimes H^q(F)$ if $\pi_1(M) = 0$.

$\vdots$

$E_\infty \simeq H^\cdot(E)$ (up to torsion).

Remark: Can include ring str. (wedge product).

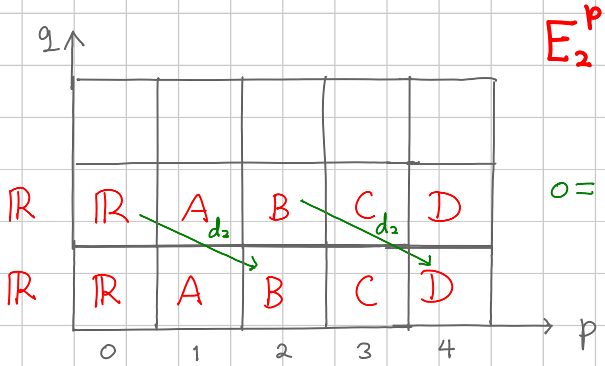
Eg. $\mathbb{CP}^2$.

$S^1 \longrightarrow S^5 \longrightarrow \mathbb{CP}^2$

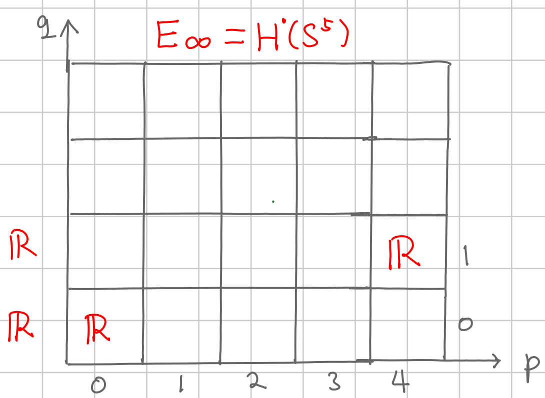
$H^\cdot(S^1) = \mathbb{R}\ \mathbb{R}$

$H^\cdot(S^5) = \mathbb{R}\ 0\ 0\ 0\ 0\ \mathbb{R}$ (degrees 0 1 2 3 4 5)

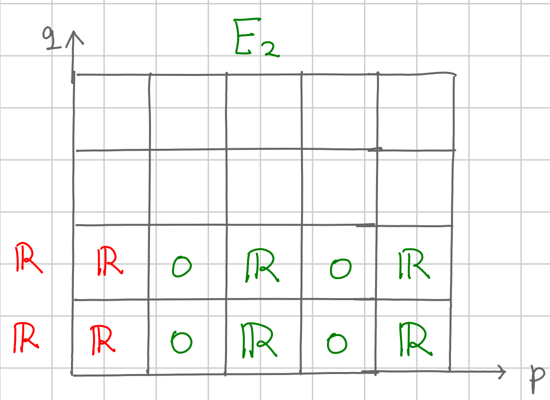
$E_2^{pq} = H^p(\mathbb{CP}^2) \otimes H^q(S^1)$



$0 = d_3 = d_4 = \ldots\ldots$



$\Longrightarrow$



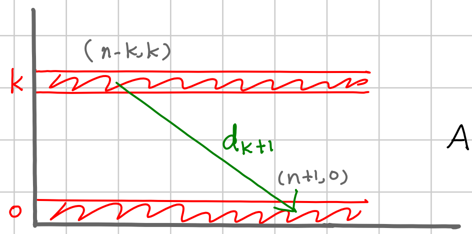
$$\Rightarrow \quad H^\bullet(\mathbb{CP}^2) = \mathbb{R} \quad 0 \quad \mathbb{R} \quad 0 \quad \mathbb{R}$$

w/ product str. incorporated, $H^\bullet(\mathbb{CP}^2) = \mathbb{R}[u]/u^3$

Similarly, $H^\bullet(\mathbb{CP}^n) = \mathbb{R}[u]/u^{n+1}$.

Gysin sequence $S^k \longrightarrow E \xrightarrow{\pi} M$ ori ($\Rightarrow$ no monodromy)

$E_2^{p,q}$
$= H^p(M) \otimes H^q(S^k)$



ALL other $d_\cdot = 0$!

$$\Rightarrow \quad 0 \to E_\infty^{n-k,k} \hookrightarrow \underbrace{E_2^{n-k,k}}_{H^{n-k}(M)} \xrightarrow{d_{k+1}} \underbrace{E_2^{n+1,0}}_{H^{n+1}(M)} \twoheadrightarrow E_\infty^{n+1,0} \longrightarrow 0$$

Also $\quad 0 \longrightarrow E_\infty^{n,0} \longrightarrow H^n(E) \longrightarrow E_\infty^{n-k,k} \longrightarrow 0$

Combine $\Rightarrow$ long exact seq.

$$\cdots \to H^n(E) \xrightarrow[\pi_*]{} H^{n-k}(M) \xrightarrow[\wedge e]{d_{k+1}} H^{n+1}(M) \xrightarrow[\pi^*]{} H^{n+1}(E) \longrightarrow \cdots$$

CH III (part 2)

$\Omega^* \rightsquigarrow H^\bullet(M,\mathbb{R})$

$H^\bullet(M,\Lambda)$ $\quad$ Λ : Abelian gp, e.g. $\mathbb{Z}, S^1, \mathbb{R}, \mathbb{C}$.
(also M : ANY topo sp.)

Singular Homology $\quad$ X any topo. space.

$S_q(X) \ni \sum_i^{\text{finite}} c_i\, s_i$ $\qquad$ $c_i \in \Lambda$ Abelian gp.
$\qquad s_i : \Delta_q \longrightarrow X$

$\partial : S_q(X) \xrightarrow{\text{bdy}} S_{q-1}(X)$ $\qquad \partial^2 = 0$

$\Rightarrow H_\bullet(X,\Lambda)$

MV seq. (Take chain supp. in $\mathcal{U}$; $S^{\mathcal{U}}_\bullet(X)$)
($\sim$ MV_{dR} w/ cpt supp.)

Singular cohomology

$S^q(X) = \mathrm{Hom}(S_q(X), \Lambda)$ w/ $d = \mathrm{adj.}(\partial)$

$\Rightarrow H^\bullet(X,\Lambda)$.

Thm. $H^\bullet_{\text{sing}}(X,\mathbb{Z}) \cong \check{H}(X,\mathbb{Z})$ if $\exists$ triangulation on X
($\Rightarrow \exists$ 'good' cover, use s.s.)

Univ. coeff. thm. ($\exists$ non-can. split)

$H_q(X;\Lambda) \simeq H_q(X)\otimes\Lambda + \mathrm{Tor}(H_{q-1}(X);\Lambda)$

$H^q(X;\Lambda) \simeq \mathrm{Hom}(H_q(X),\Lambda) + \mathrm{Ext}(H_{q-1}(X),\Lambda)$

In particular, $/\mathbb{Z}$

$$\begin{cases} \text{Free } H^q \simeq \text{Free } H_q \\ \text{Tor } H^q \simeq \text{Tor } H_{q-1} \end{cases}$$

Tor / Ext

$\forall A$ Abelian gp.

$$0 \to \underset{\text{relation}}{R} \longrightarrow \underset{\text{gen.}}{F} \longrightarrow A \longrightarrow 0 \qquad \text{free resol}^{\underline{n}}.$$

$$\rightsquigarrow \quad 0 \to \mathrm{Hom}(A,\Lambda) \to \mathrm{Hom}(F,\Lambda) \to \mathrm{Hom}(R,\Lambda) \to \mathrm{Ext}(A,\Lambda) \to 0$$

$$0 \to \mathrm{Tor}(A,\Lambda) \to R \otimes \Lambda \to F \otimes \Lambda \to A \otimes \Lambda \to 0$$

Ext	$\mathbb{Z}$	$\mathbb{Z}_n$
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	$\mathbb{Z}_m$	$\mathbb{Z}_{(m,n)}$

Tor	$\mathbb{Z}$	$\mathbb{Z}_n$
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	0	$\mathbb{Z}_{(m,n)}$

Ex. Use $U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1}$ & s.s. to compute $H^\bullet(U(n))$.

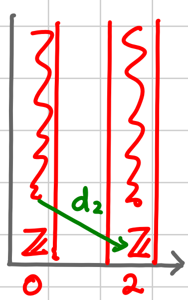
Path fibration (serre fibration)

$p \in X$

$$\Omega_{p\to q} X \longrightarrow \underset{\text{Contractible}}{P_p X} \xrightarrow{ev._{t=1}} X$$

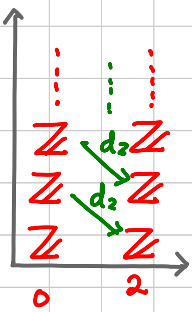
Eg. $X = S^2 \Rightarrow H^q(\Omega S^2) = \mathbb{Z} \quad \forall q$

reason:



trivial S^1

$\Rightarrow$



$E_2 \Rightarrow E_\infty$ $\qquad$ E_2

Similarly, $H^\bullet(\Omega S^{2n}) \cong \dfrac{\mathbb{Z}[x]}{x^2} \otimes \mathbb{Z}[e]$ $\qquad$ $\deg x = 2n-1$, $\deg e = 2(2n-1)$

$H^\bullet(\Omega S^{2n+1}) \cong \mathbb{Z}[u] \qquad \deg u = 2n$

Homotopy groups $\pi_q(X, *) \ni \alpha : (S^q, \cdot) \to (X, *)$ up to homotopy.

- group str.
- Abelian if $q \geqslant 2$
- Hard to compute. $\nexists$ MV arguments (Van Kampen for π_1)
- $F \to E \to B$ fiber bdl $\Rightarrow$ long exact seq.

$$\cdots \to \pi_q(F) \to \pi_q(E) \to \pi_q(B) \to \pi_{q-1}(F) \to \cdots\cdots$$

Simplest space S^n ; $\pi_q(S^n) = ?$

- $\begin{cases} \pi_1(S^1) = \mathbb{Z} \\ \pi_{>1}(S^1) = 0 \end{cases}$ (use $\mathbb{Z} \to \overset{*}{\underset{S}{\mathbb{R}}} \to S^1$)
- $\pi_{q-1}(\Omega X) \simeq \pi_q(X)$ $\forall\, q \geqslant 2$

- $\pi_{<n}(S^n) = 0$ (perturb & miss north, shrink to south)

  $\pi_n(S^n) \xrightarrow[\simeq]{\deg} \mathbb{Z}$ (need to cancel $+1, -1$ pair. – nontrivial topo. arguments.)
- $\exists$ natural $i : \pi_q(X) \to H_q(X, \mathbb{Z})$

Hurwicz Theorem. X path-conn. topo sp.

(i) $H_1(X, \mathbb{Z}) = \pi_1(X) / [\pi_1, \pi_1]$ Abelianization

(ii) If X CW cpx. & $\pi_1 = 0$,

$\pi_{<n} = 0 \Rightarrow \pi_n \cong H_n$.

$\pi_{n+1} \twoheadrightarrow H_{n+1}$

Cor $\pi_n(S^n) \simeq \mathbb{Z}$.

Consider $\Omega X \to PX \to X$ path fibration

$$E_2: \begin{array}{|lll} H_1(\Omega X) & \nwarrow & \\ \mathbb{Z} & \underset{\because \pi_1 = 0}{0} & H_2(X) \\ \hline \end{array} \quad \Rightarrow \quad H_2(X) \simeq H_1(\Omega X) \quad (\because PX \sim *)$$

$$\simeq \pi_1(\Omega X) \quad (\because \pi_1(\Omega X) \text{ Abelian } \& \text{ (i)})$$

$$\simeq \pi_2(X)$$

Similar for $n \geq 2$.

How about $\pi_{>n}(S^n)$?

Hopf Invariant $\pi_3(S^2) \simeq \mathbb{Z}$

Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow \mathbb{CP}^1 \simeq S^2$

$$\Rightarrow \quad \cdots \to \pi_q(S^1) \longrightarrow \pi_q(S^3) \longrightarrow \pi_q(S^2) \longrightarrow \pi_{q-1}(S^1) \longrightarrow \cdots.$$

$$\Rightarrow \left(\because \begin{array}{l} \pi_1 S^1 = \mathbb{Z} \\ \pi_{>1} S^1 = 0 \end{array} \right) \quad \pi_q(S^3) \simeq \pi_q(S^2) \text{ for } q \geq 3 \quad \Rightarrow \quad \pi_3(S^2) \simeq \mathbb{Z}$$

- Describe $H : \pi_3(S^2) \xrightarrow{\simeq} \mathbb{Z}$

$[f] \in \pi_3(S^2)$; $f: S^3 \longrightarrow S^2$ C^∞

Take generator $[\alpha] \in H^2_{dR}(S^2)$ $\quad \alpha = \frac{1}{4\pi}(u_1 du_2 du_3 - u_2 du_1 du_3 + u_3 du_1 du_2)$

$du_1 du_2 du_3 = 4\pi \frac{dr}{r} \wedge \alpha$ on $\mathbb{R}^3$

$$[f^*\alpha] \in H^2(S^3) = 0$$

$$\Rightarrow \quad f^*\alpha = d\omega \quad \exists\, \omega \in \Omega^1(S^3)$$

$$H(f) = \int_{S^3} \omega \wedge d\omega$$

- $H(\text{Hopf fibrat}^{\underline{n}}) = 1$
- $H(f) =$ Linking # between $f^{-1}(a)$ & $f^{-1}(b)$.
- Generalize $H : \pi_{4n-1}(S^{2n}) \xrightarrow{\cong} \mathbb{Z}$

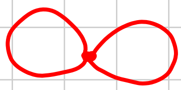
  Fact : All other $\pi_\cdot(S^m)$ are torsion.

Eilenberg-MacLane spaces (Building blocks)

- $K(A, n)$ w/ $\begin{cases} \pi_n \simeq A \\ \pi_{\neq n} = 0 \end{cases}$ ($\exists !$ $\because$ adding handle)

- $H^n(X, A) \simeq [X, K(A, n)]$ (Obstruction theory)

Eg

- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z} * \mathbb{Z}, 1) =$ 

- Curv. $(M) \leq 0 \Rightarrow M \simeq K(\pi, 1)$ $(\because \tilde{M} = \mathbb{R}^m)$
  e.g. $g \geq 1$ Riemann surface

- $\Omega K(A, n) = K(A, n-1)$ $(\because \pi_q(\Omega X) = \pi_{q+1}(X))$

- $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$ $\left(\mathbb{Z}_2 \to \underset{\underset{*}{\wr}}{S^\infty} \to \mathbb{R}P^\infty\right)$

- $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ $\left(S^1 \to \underset{\underset{*}{\wr}}{S^\infty} \to \mathbb{C}P^\infty\right)$
  s.s. $\Rightarrow H^\bullet(\mathbb{C}P^\infty) \simeq \mathbb{Z}[x]$

- Similar for ∞ Lens spaces $L(\infty, q) = K(\mathbb{Z}_q, 1)$
  Def$^{\underline{n}}$: $\mathbb{Z}_q \leq S^1 \curvearrowright S^{2n+1}$; $S^{2n+1}/S^1 = \mathbb{C}P^n$; $S^{2n+1}/\mathbb{Z}_q =: L(n, q)$

  s.s. $\Rightarrow H^\bullet(L(n, q)) = \begin{cases} \mathbb{Z} & \bullet = 0, 2n+1 \\ \mathbb{Z}_q & \bullet = 2, 4, \dots, 2n \\ 0 & \text{otherwise} \end{cases}$

  In particular, $H^\bullet(\underbrace{L(\infty, q)}_{K(\mathbb{Z}_q, 1)})_{\mathbb{Q}} = \mathbb{Q}$

  $\Rightarrow H^\bullet(K(G, n))_{\mathbb{Q}} = \mathbb{Q}$ $\forall G$ torsion Abelian group.

- $H^\bullet(K(\mathbb{Z},n))_{\mathbb{Q}} = \Lambda[x]$ $\quad \deg x = n$
  (graded comm. alg.)

  (Use s.s. & $K(\mathbb{Z},n-1) \to \underset{\simeq *}{PK(\mathbb{Z},n)} \to K(\mathbb{Z},n)$)

- $H^q(K(\mathbb{Z},3)) = \underset{1}{\mathbb{Z}} \; 0 \; 0 \; \underset{s}{\mathbb{Z}} \; 0 \; 0 \; \underset{s^2}{\mathbb{Z}_2} \; 0 \; \underset{t}{\mathbb{Z}_3} \; \cdots\cdots$

Postnikov Tower.

Theorem : $\forall$ CW cpx X , $\forall\, n$

$\exists \quad X \longrightarrow Y_n \longrightarrow \cdots\cdots \longrightarrow Y_3 \xrightarrow{K(\pi_3,3)} Y_2 \xrightarrow{K(\pi_2,2)} Y_1 \xrightarrow{K(\pi_1,1)} \bullet$

$\cdots$ fibration

induces $\quad \pi_{\leq n-1}(X) \xrightarrow{\cong} \pi_{\leq n-1}(Y_n)$

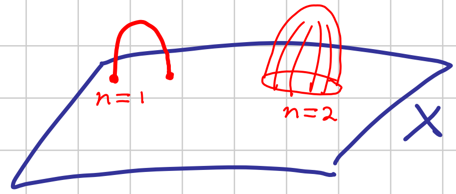
i.e. $\quad X$ is a "twist" product $\prod_{n=1}^{\infty} K(\pi_n(X), n)$.

Tool : Attaching cell

$[f : S^{n-1} \to X] \in \pi_{n-1}(X)$

$$X_f := X \cup_f e^n = X \amalg e^n / f(u) \sim u \quad \forall\, u \in S^{n-1}$$

$$\begin{array}{c} X_f \\ \uparrow \\ X \end{array}$$



Prop. $\pi_{<n-1}(X) \cong \pi_{<n-1}(X_f)$

$\pi_{n-1}(X) \twoheadrightarrow \pi_{n-1}(X_f)$

Prop. $H_{\neq (n-1), n}(X) \cong H_{\neq (n-1), n}(X_f)$

$$0 \to H_n(X) \to H_n(X_f) \to \mathbb{Z} \xrightarrow{f_*} H_{n-1}(X) \to H_{n-1}(X_f) \to 0 \qquad \text{(MV)}$$

Constr. P. tower:

X • $[f] \in \pi_{n+1}(X) \rightsquigarrow X_f$ kill it (may change $\pi_{\geq n+2}$ but preserve $\pi_{\leq n}$)

Kill them off (no π_{n+1})

Then kill $\pi_{n+2}(X), \pi_{n+3}(X), \ldots\ldots$

$\Rightarrow Y_n$ $\left(\begin{array}{l} \pi_{\geq n+1} = 0 \\ \pi_{\leq n}(Y_n) = \pi_{\leq n}(X) \end{array}\right)$

• Kill $\pi_{\geq n}(Y_n) \Rightarrow Y_{n-1} \supseteq Y_n$

• $X \subset Y_n \subset Y_{n-1} \subset \ldots \subset Y_1$

• "Convert" to fibration

$X \to Y_n \to Y_{n-1} \to \ldots \to Y_1$

fiber $= K(\pi_q(X), q)$ ($\because$ homotopy seq. of π. for fibering.)

Remark: Up to homotopy, every map $f: X \to Y$ is an inclusion / fibering.

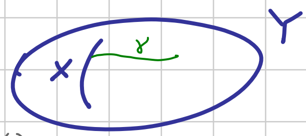
- Inclusion:

$$f: X \longrightarrow Y$$
$$\wr\ \text{h.e.}$$
$$X \longrightarrow (X \times [0,1]) \cup Y / (x,1) \sim f(x)$$
$$x \longmapsto (x, 0)$$

- Fibering (say $f: X \subset Y$)

$$X \overset{\text{h.e.}}{\sim} P_X Y \ni \begin{array}{l} \gamma: [0,1] \to Y \\ \gamma(0) \in X \end{array}$$

(can shrink back to X)



$$P_X Y \longrightarrow Y \qquad \text{fiber} \sim \Omega Y$$
$$\gamma \longmapsto \gamma(1)$$

homotopy to f.

$\pi_4(S^3) \simeq \mathbb{Z}_2$

$S^3 \subseteq Y_4$ by attaching cell of dim ≥ 6 ($\because$ Kill $\pi_{\geq 5}$)

i.e. $Y_4 = S^3 \cup e^6 \cup \cdots\cdots$

$\Longrightarrow H_4(Y_4) = 0 = H_5(Y_4)$

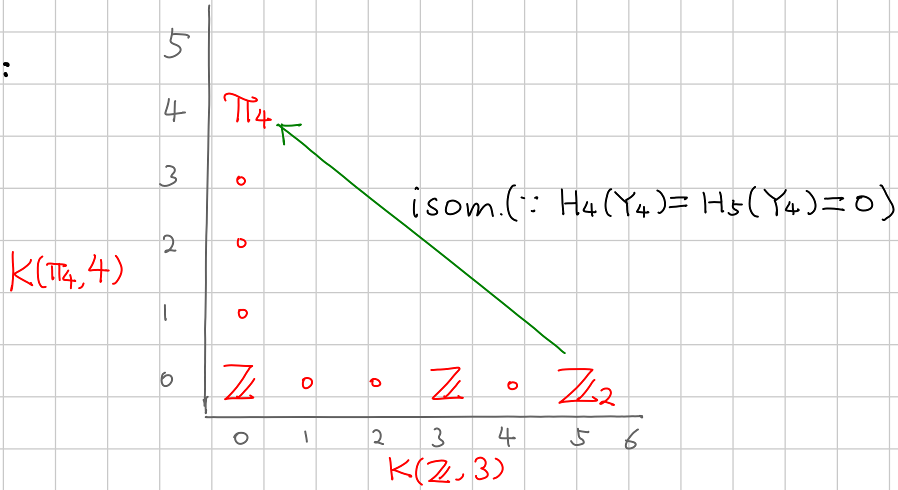
$$0 \to H_6(X) \to H_6(X_f) \to \mathbb{Z} \xrightarrow{f_*} \cancel{H_5(X)} \to H_5(X_f) \to 0$$
$$\qquad\ \ \| \qquad\qquad\qquad\qquad\qquad\qquad\ \ \|$$
$$\qquad\ \ S^3 \qquad\qquad\qquad\qquad\qquad\qquad S^3$$

$\underline{\text{P. tower}} \Longrightarrow$ fibrat^{n}: $K(\pi_4, 4) \to Y_4 \longrightarrow \underbrace{Y_3}_{K(\mathbb{Z},3)}$ $\left(\because \begin{array}{l} \pi_{<3}(S^3) = 0 \\ \pi_3(S^3) = \mathbb{Z} \end{array}\right)$

Recall $H^q(K(\mathbb{Z},3)) = \underset{1}{\mathbb{Z}}\ 0\ 0\ \underset{s}{\mathbb{Z}}\ 0\ \underset{s^2}{\mathbb{Z}_2}\ 0\ \underset{t}{\mathbb{Z}_3} \cdots\cdots$

Homology s.s.:

E_2.



$\Rightarrow \quad \pi_4(S^3) = \mathbb{Z}_2$

Will show later $\pi_5(S^3) \simeq \mathbb{Z}_2$.

Whitehead Tower

P. tower: Kill $\pi_{>n}$
W. tower: Kill $\pi_{<n}$

Thm. $\exists$ fibrations

$$\cdots \longrightarrow X_n \xrightarrow[K(\pi_n, n-1)]{} X_{n-1} \longrightarrow \cdots \xrightarrow[K(\pi_2, 1)]{} X_1 \xrightarrow[K(\pi_1, 0)]{} X$$

s.t. $\quad \pi_{<n}(X_n) = 0$

$\quad \pi_{\geq n}(X_n) \xrightarrow{\cong} \pi_{\geq n}(X)$

Note: $X_1 = \widetilde{X}$

Constr: Use path space instead of attach cell.

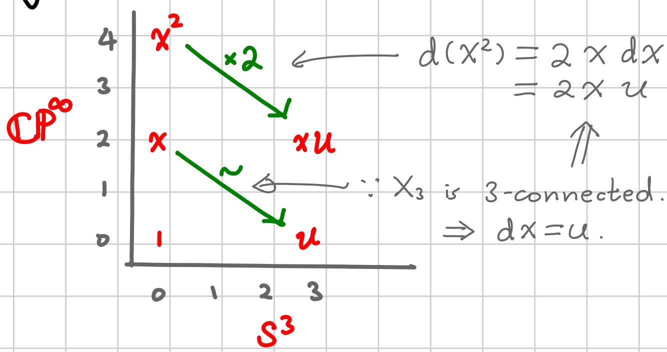
$\pi_5(S^3) \simeq \mathbb{Z}_2$

W. tower: $X_4 \xrightarrow[K(\pi_4,3)]{} X_3 \xrightarrow[K(\mathbb{Z},2)]{} X_2 = X_1 = S^3$

$$\pi_5(S^3) \simeq \pi_5(X_4) \underset{\text{Hurwicz}}{\simeq} H_5(X_4) = ?$$

Coh. s.s. for $\mathbb{C}P^\infty \to X_3 \to S^3$

E_2:

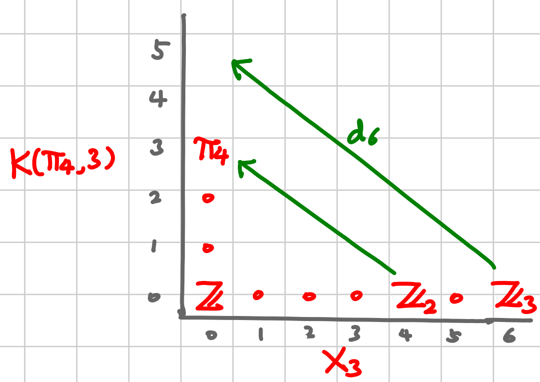


$$\Longrightarrow \quad H^q(X_3) = \overset{0}{\mathbb{Z}}\ \overset{1}{0}\ \overset{2}{0}\ \overset{3}{0}\ \overset{4}{0}\ \overset{5}{\mathbb{Z}_2}\ \overset{6}{0}\ \overset{7}{\mathbb{Z}_3}\ \overset{8}{0}\ \overset{9}{\mathbb{Z}_4}\ \overset{10}{0}\ \overset{11}{\mathbb{Z}_5}$$

$$\xrightarrow{\text{univ. coeff. thm.}} \quad H_q(X_3) = \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}_2\ 0\ \mathbb{Z}_3\ 0\ \mathbb{Z}_4\ 0\ \mathbb{Z}_5\ 0$$

Homology s.s. for

$K(\pi_4,3) \longrightarrow X_4 \longrightarrow X_3$



$\pi_4(S^3) \simeq \mathbb{Z}_2$ $\quad (\because X_4 : 4\text{-connected})$

Recall: For $K(\mathbb{Z}_2, 3)$, $H_4 = 0$ & $H_5 \simeq \mathbb{Z}_2$

$\Rightarrow d_6 : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2 \quad \Rightarrow$ zero map.

$\Rightarrow \underbrace{H_5(X_4)}_{\pi_5(S^3)} \simeq \mathbb{Z}_2$

QED.

Another appl. of W. tower:

Serre thm: (1) n odd $\Rightarrow \pi_q(S^n)_{\mathbb{Q}} = 0$ except π_n

(2) n even $\Rightarrow \pi_q(S^n)_{\mathbb{Q}} = 0$ except π_n, π_{2n-1}

(Hopf map).

Minimal models

$\pi_\bullet(X)$ Very hard

$\pi_\bullet(X)_{\mathbb{Q}}$ Much easier (from $(\Omega^\bullet, \wedge, d)$)

$(A^\bullet, \wedge, d)$ diff. gr. comm. alg. $/\mathbb{R}$.

Def: $\mathcal{M}$ min. model for A if

(i) $\mathcal{M}$ free, (ii) $\exists\ f: \mathcal{M} \xrightarrow{\text{q-isom.}} A$, $\cong$ on $H^\bullet$

(iii) If $x \in \mathcal{M}$ generator, then

$dx \in \mathcal{M}^+ \wedge \mathcal{M}^+$ $\quad (\mathcal{M}^+ = \bigoplus_{i>0} \mathcal{M}^i)$

(call decomposable).

- A 1-connected (i.e. $H^0 = \mathbb{R}$, $H^1 = 0$)

$\Rightarrow \exists$ "!" m.m. $\mathcal{M}$

(Deligne-Griffiths-Morgan-Sullivan)

Thm. $\pi_1 M = 0$ + $\mathcal{M}$ = m.m. of $(\Omega^\bullet(M), \wedge, d)$

$\Rightarrow$ $\dim \pi_q(M)_{\mathbb{Q}}$ = # of generators of $\mathcal{M}$ in dim q.

Thm. If M Kähler, then $(H^\bullet(M), \wedge)$ determines $\mathcal{M}$ (thus $\mathbb{Q}$-homotopy type of M).

Eg. $\dim \pi_q(S^2 \vee S^2)_{\mathbb{Q}} = 0 \ 2 \ 3 \ 2 \ 3 \ 6 \ \ldots\ldots$

Example: • $H^\bullet(S^{2n-1}) = \Lambda[x] \Rightarrow$ m.m.

• $H^\bullet(S^{2n}) = \mathbb{R}[a]/a^2$ NOT free

$\mathcal{M} = \Lambda(x, y)$ $\quad \dim x = 2n, \ \dim y = 4n-1 \quad dx = 0, \ dy = x^2$

$$f : \mathcal{M} \longrightarrow \Omega^\bullet(S^{2n})$$
$$x \mapsto \mathrm{Vol}_{S^{2n}}$$
$$y \mapsto 0$$

• $H^\bullet(\mathbb{CP}^n) = \mathbb{R}[x]/x^{n+1}$

$\mathcal{M} = \Lambda(x, y)$ $\quad \dim x = 2, \ \dim y = 2n+1 \quad dx = 0, \ dy = x^{n+1}$

Chapter 4. Characteristic classes

Recall: Euler class

ori. $\mathbb{R}^n \longrightarrow E \longrightarrow M \quad \rightsquigarrow \quad e(E) \in H^n(M)$

• Naturality

$$e(f^*E) = f^* e(E)$$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\ X \arrow[r, "f"] & M \end{tikzcd}$$

$(f^*E)_x := E_{f(x)}$

(reason: fiber $\rho\, dx^1 \wedge \cdots \wedge dx^n$ $\mathbb{R}^n$ + closed $\Big\{ \Phi|_0 = e \Big)$
$\Rightarrow$ naturality for Thom class

• $e(E_1 \oplus E_2) = e(E_1) \wedge e(E_2)$

(same reason)

More classes for $\mathbb{C}^n \to E \to M$

- $U(n) \subset SO(2n) \Rightarrow$ oriented

Can consider $e(\Lambda^k E)$ or $e(\mathrm{Sym}^k E)$

Ranks are very large except $\Lambda^{\mathrm{TOP}} E$, a line bundle

($\Lambda^{n-1} E = (\Lambda^n E) \otimes E^*$, ess. the same.)

If $E = L_1 + L_2 + \dots + L_n$, sum of $\mathbb{C}$-line bundles

$e(E) = e(L_1) \cdot e(L_2) \cdots\cdots e(L_n)$

$$e(\Lambda^n E) = e(L_1 \otimes L_2 \otimes \dots \otimes L_n)$$
$$= e(L_1) + e(L_2) + \cdots\cdots + e(L_n)$$

$\left(\because\ e(L) \text{ on } U_\alpha \cong \frac{i}{2\pi} \sum_\gamma d(\rho_\gamma \, d \log g_{\gamma\alpha}) \right)$

Write $c_k(E) :=$ k^{th} elem. symm. polyn. in $e(L_i)$'s.

i.e. $$c(E) = 1 + c_1(E) + \cdots\cdots\cdots + c_n(E)$$
$$= \prod_{i=1}^n \underbrace{(1 + e(L_i))}_{1 + c_1(L_i)}$$

i.e. $$c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$$

- $c_1(E) = \sum_i c_1(L_i) = e(\Lambda^n E)$

$c_n(E) = \prod_i c_1(L_i) = e(E)$

$c_{>n}(E) = 0$

Indeed $c(E)$ can be defined $\forall$ $\mathbb{C}$-vector bundle E satisfying (uniqueness ✓)

(i) same as before if E = Sum of line bdls

(ii) naturality $c(f^*E) = f^* c(E)$

(iii) Whitney product formula, $c(E_1 + E_2) = c(E_1) \cdot c(E_2)$

2 Approaches:

(1) Abelianization $\quad \forall\ \mathbb{C}^n \to E \to M,\ \exists\ f : X \to M$

s.t. $\quad f^*E = L_1 + L_2 + \dots + L_2$ over X

& $\quad f^* : H^\bullet(M) \hookrightarrow H^\bullet(X) \qquad$ ($\Rightarrow$ splitting principle)

(2) Classifying space $\quad \exists$ Univ. bdl. $\mathbb{C}^n \to \mathcal{E} \to \mathbb{B}$

$\forall\ \mathbb{C}^n \to E \to M,\ \exists\ f : X \to \mathbb{B}$

s.t. $\quad E = f^*\mathcal{E} \qquad$ (only use naturality).

Defining Chern classes

Recall: For $V \simeq \mathbb{C}^n$

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] & V \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\ & & \mathbb{P}(V) \simeq \mathbb{P}^{n-1} & & \end{tikzcd}$$

$$\begin{tikzcd} (v \in \ell) \in S \arrow[r, "\text{blowup at origin}"] \arrow[d] & v \in V \\ \ell \in \mathbb{P}(V) & \end{tikzcd}$$

$$H^\bullet(\mathbb{P}^{n-1}) \simeq \mathbb{R}[x]/x^n \quad \text{w/} \quad x = e(S^*)$$

For family $\mathbb{C}^n \longrightarrow E \longrightarrow M$

$\rightsquigarrow \quad \mathbb{P}^{n-1} \longrightarrow \mathbb{P}(E) \longrightarrow M$

Leray-Hirsch $\Rightarrow$ $\quad (\simeq H^\cdot(M) \otimes H^\cdot(\mathbb{P}^{n-1})$ as vector sp$)$

$H^\bullet(\mathbb{P}(E)) \simeq H^\cdot(M)[x] \,/\, x^n + \text{l.o.t.}$ as $H^\cdot(M)$-mod.

where $x = e(\mathcal{S}^*)$,

$$\begin{tikzcd} & & 0 \arrow[r] & \mathcal{S} \arrow[r] & \pi^*E \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\ E \arrow[d] & & & & \mathbb{P}(E) \arrow[llll, "\pi"'] & & \\ M & & & & & & \end{tikzcd}$$

In particular, $\pi^* : H^\cdot(M) \hookrightarrow H^\bullet(\mathbb{P}(E))$.

Indeed, the relation is given by

$$x^n + c_1(E)\, x^{n-1} + c_2(E)\, x^{n-2} + \dots + c_n(E) = 0$$

with $c_r(E) \in H^{2r}(M)$. $\quad (\pi^*E \simeq S + Q$ partially split$)$.

$$\begin{tikzcd} E \arrow[d] & \pi^*E = \overset{\text{rk}\,1 \checkmark}{S_1} + Q_1 \arrow[d] & \pi^*Q = \overset{\text{rk}\,1 \checkmark}{S_2} + Q_2 \arrow[d] & & \pi_\cdot^*E = S_1 + S_2 + \dots + S_n \arrow[d] \\ M & \mathbb{P}_M(E) = M_1 \arrow[l, "\pi"'] & \mathbb{P}_{M_1}(Q_1) = M_2 \arrow[l, "\pi_1"'] \arrow[r, dashed, "\text{total } \underset{\text{rk}\,E}{\underbrace{n}} \text{ step.}"] & & M_n \end{tikzcd}$$

sum of line bdl

Also $\quad H^\cdot(M) \overset{\pi_\cdot^*}{\hookrightarrow} H^\cdot(M_n)$

$\Rightarrow$ Can verify all properties.

Remark: Can do all steps at once.

$$\mathbb{P}^{n-1} = \{0 \subset W_1 \subset \mathbb{C}^n\} \qquad \dim W_i = i$$

$$= GL(n,\mathbb{C}) \Big/ \begin{array}{c} \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array} \begin{array}{l} {\scriptstyle 1} \\ {\scriptstyle n-1} \end{array}$$

$$= U(n)/U(1)U(n-1) \qquad \text{(by choosing metric)}$$

$$= \{ \mathbb{C}^n = W_1 \oplus W_1^\perp \}$$

Grassmannian

$$Gr(k,n) = \{0 \subset W_k \subset \mathbb{C}^n\} \simeq GL(n,\mathbb{C}) \Big/ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}^{k}_{n-k}$$

$$= \{\mathbb{C}^n = W_k \oplus W_k^{\perp}\} \simeq U(n)/U(k)U(n-k)$$

Flag variety

$$Fl_n = \{0 \subset W_1 \subset W_2 \subset \dots \subset W_n = \mathbb{C}^n\} \simeq GL(n,\mathbb{C}) \Big/ \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$$

$$= \{\mathbb{C}^n \overset{\perp}{=} S_1 + S_2 + \dots + S_n\} \simeq U(n)/U(1)^n$$

$$W_k = S_1 + S_2 + \dots\dots + S_k .$$

Family: $\quad \mathbb{C}^n \longrightarrow E \longrightarrow M$

$\rightsquigarrow \quad Fl_n \longrightarrow Fl_n(E) \xrightarrow{\pi} M$

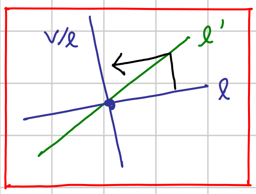
w/ $\quad \pi^* E \overset{\mathcal{S}_n}{\supset} \mathcal{W}_{n-1} \overset{\mathcal{S}_{n-1}}{\supset} \mathcal{W}_{n-2} \overset{\mathcal{S}_{n-2}}{\supset} \dots \overset{\mathcal{S}_2}{\supset} \mathcal{W}_1 \overset{\mathcal{S}_1}{\supset} 0$

Topo. $\quad \pi^* E = \mathcal{S}_1 + \mathcal{S}_2 + \dots + \mathcal{S}_n \quad$ sum of line bdls.

Proj. space / Grassmannian / Flag variety

$$\frac{U(n)}{U(1)\,U(n-1)} \qquad \frac{U(n)}{U(k)U(n-k)} \qquad \frac{U(n)}{U(1)^n}$$

$$T_{\mathbb{P}(V)} \simeq Hom(S, Q)$$



V

$$\Downarrow$$

$$c(T_{\mathbb{P}^{n-1}}) = c(S^* \otimes Q)$$

$$\overset{\text{Whitney}}{=} c(S^* \otimes Q + S^* \otimes S)$$

$$\overset{\text{Tauto. seq.}}{=} c(S^* \otimes \mathbb{C}^n) \overset{\text{Whitney}}{=} \prod^n c(S^*)$$

$$= (1+x)^n$$

For $\quad \mathbb{C}^n \longrightarrow E \longrightarrow M$

$$H^{\bullet}(\mathbb{P}(E)) = H^{\bullet}(M)[x] \big/ x^n + c_1(E)x^{n-1} + \dots + c_n(E) \quad \text{w/ } x = c_1(S^*)$$

$$= H^{\bullet}(M)[c(S), c(Q)] \big/ c(S)\cdot c(Q) = \pi^* c(E)$$

$$H^{\bullet}(Fl_n(E)) = H^{\bullet}(M)[x_1, \dots, x_n] \Big/ \prod_{i=1}^{n}(1 + x_i) = c(E)$$

$$\text{w/ } \quad x_i = c_1(\mathcal{S}_i)$$

(Omit: Classifying spaces).